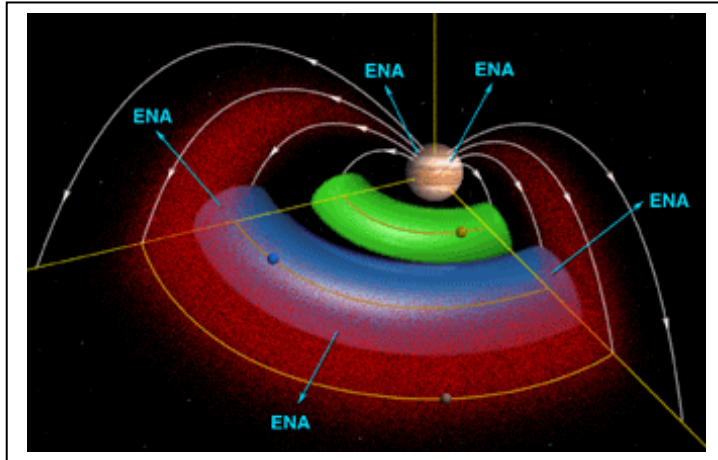


The Io Plasma Torus



The satellite of Jupiter, Io, is a volcanically active moon that ejects 1,000 kilograms of ionized gas into space every second. This gas forms a torus encircling Jupiter along the orbit of Io. We will estimate the total mass of this gas based on data from the NASA Cassini and Galileo spacecraft.

Image: Io plasma torus (Courtesy NASA/Cassini)

Problem 1 - Galileo measurements obtained in 2001 indicated that the density of neutral sodium atoms in the torus is about 35 atoms/cm^3 . The spacecraft also determined that the inner boundary of the torus is at about 5 R_J , while the outer boundary is at about 8 R_J . (1 $R_J = 71,300 \text{ km}$). A torus is defined by the radius of the ring from its center, R , and the radius of the circular cross section through the donut, r . What are the dimensions, in kilometers, of the Io torus based on the information provided by Galileo?

Problem 2 - Think of a torus as a curled up cylinder. What is the general formula for the volume of a torus with radii R and r ?

Problem 3 - From the dimensions of the Io torus, what is the volume of the Io torus in cubic meters?

Problem 4 - From the density of sodium atoms in the torus, what is A) the total number of sodium atoms in the torus? B) If the mass of a sodium atom is 3.7×10^{-26} kilograms, what is the total mass of the Io torus in metric tons?

Advanced Math:

Problem 5 - Using the 'washer method' in integral calculus, derive the formula for the volume of a torus with a radius equal to R , and a cross-section defined by the formula $x^2 + y^2 = r^2$. The torus is formed by revolving the cross section about the Y axis.

Answer Key

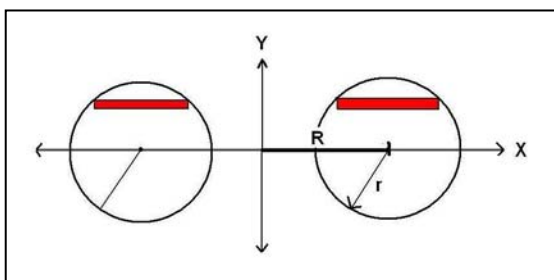
Problem 1 - The mid point between 5 Rj and 8 Rj is $(8+5)/2 = 6.5$ Rj so $R = 6.5$ Rj and $r = 1.5$ Rj. Then $R = 6.5 \times 71,300$ so **$R = 4.6 \times 10^5$ km**, and $r = 1.5 \times 71,300$ so **$r = 1.1 \times 10^5$ km**.

Problem 2 - The cross-section of the cylinder is πr^2 , and the height of the cylinder is the circumference of the torus which equals $2 \pi R$, so the volume is just $V = (2 \pi R) \times (\pi r^2)$ or **$V = 2 \pi^2 R r^2$** .

Problem 3 - Volume = $2 \pi^2 (4.6 \times 10^5 \text{ km}) (1.1 \times 10^5 \text{ km})^2$ so **$V = 1.1 \times 10^{17} \text{ km}^3$** .

Problem 4 - A) $35 \text{ atoms/cm}^3 \times (100000 \text{ cm/1 km})^3 = 3.5 \times 10^{16} \text{ atoms/km}^3$. Then number = density \times volume so $N = (3.5 \times 10^{16} \text{ atoms/km}^3) \times (1.1 \times 10^{17} \text{ km}^3)$, so **$N = 3.9 \times 10^{33} \text{ atoms}$** .
B) The total mass is $M = 3.9 \times 10^{33} \text{ atoms} \times 3.7 \times 10^{-20} \text{ kilograms/atom} = 1.4 \times 10^{14} \text{ kilograms}$. 1 metric ton = 1000 kilograms, so the total mass is **$M = 100 \text{ billion tons}$** .

Advanced Math:



$$V = 8 \pi R \int_0^r (r^2 - y^2)^{1/2} dy$$

Recall that the volume of a washer is given by $V = \pi (R(\text{outer})^2 - R(\text{inner})^2) \times \text{thickness}$. For the torus figure above, we see that the thickness is just dy . The distance from the center of the cross section to a point on the circumference is given by $r^2 = x^2 + y^2$. The width of the washer (the red volume element in the figure) is parallel to the X-axis, so we want to express its length in terms of y , so we get $x = (r^2 - y^2)^{1/2}$. The location of the outer radius is then given by $R(\text{outer}) = R + (r^2 - y^2)^{1/2}$, and the inner radius by $R(\text{inner}) = R - (r^2 - y^2)^{1/2}$. We can now express the differential volume element of the washer by $dV = \pi [(R + (r^2 - y^2)^{1/2})^2 - (R - (r^2 - y^2)^{1/2})^2] dy$. This simplifies to $dV = \pi [4R (r^2 - y^2)^{1/2}] dy$ or $dV = 4 \pi R (r^2 - y^2)^{1/2} dy$. The integral can immediately be formed from this, with the limits $y = 0$ to $y=r$. Because the limits to y only span the upper half plane, we have to double this integral to get the additional volume in the lower half-plane. The required integral is shown above.

This integral can be solved by factoring out the r from within the square-root, then using the substitution $U = y/r$ and $dU = 1/r dy$ to get the integrand $dV = 8 \pi R r^2 (1 - U^2)^{1/2} dU$. The integration limits now become $U=0$ to $U=1$. Since r and R are constants, this is an elementary integral with the solution $V = 1/2 U (1 - U^2)^{1/2} + 1/2 \arcsin(U)$. When this is evaluated from $U=0$ to $U=1$, we get

$$V = 8 \pi R r^2 [0 + 1/2 \arcsin(1)] - [0 + 1/2 \arcsin(0)]$$

$$V = 8 \pi R r^2 1/2 (\pi/2)$$

$$\mathbf{V = 2 \pi R r^2}$$